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Fatigue life under along-wind loading — closed-form solutions

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Abstract

Fatigue damage for structures under along-wind loading is considered in this paper. After making certain simplifying assumptions, closed-form expressions for upper and lower limits on fatigue life are derived. Both narrow band resonant response and wide-band background response are considered. An example of calculation of fatigue life is given.

Keywords: Along-wind; Background; Fatigue; Resonant; Wind loading

1. Introduction

The fluctuating nature of wind loading produces fluctuating stresses in structures with contributions from resonant and background (sub-resonant) components. The proportion of each depends on the natural frequencies and damping of the structure, including aerodynamic damping. For most structures the wide-band background contribution is usually dominant, when along-wind loading is the principal source of excitation. Although fatigue failures under wind loading (Fig. 1) have been rare, because of the fluctuating nature of the loading and response, fatigue may need to be considered in the design of steel and aluminium structures that have high exposure to turbulent wind loads. This problem has been considered previously (e.g. Patel and Freathy [1]), and has usually resulted in methods that require extensive calculation.

In this paper, a 'closed-form' solution for fatigue damage and fatigue life of structures subjected to along-wind loading is derived. Certain simplifying assumptions have been made to generate closed-form solutions, but the errors resulting from these assumptions are probably less than those resulting from the uncertainties in the input parameters in practical design situations. The results may be used for initial design calculations to determine whether fatigue under wind loading is in fact likely to be a problem.

2. Fatigue failure models

The 'fatigue' of metallic materials under cyclic loading has been well researched, although the treatment of fatigue damage under the random dynamic loading characteristic of wind loading is less well developed.

In the usual failure model for the fatigue of metals it is assumed that each cycle of a sinusoidal stress response inflicts an increment of damage which depends on the amplitude of the stress. Each successive cycle then generates additional damage which accumulates in proportion to the number of cycles until failure occurs. The results of constant amplitude fatigue tests are usually expressed in the form of an *s–N* curve, where *s* is the stress amplitude, and *N* is the number of cycles until failure. For many materials, the *s–N* curve is well approximated by a straight line when log *s* is plotted against log *N*. This implies an equation of the form:

$$Ns^m = K \tag{1}$$

where K is a constant which depends on the material, and the exponent m varies between about 5 and 20.

A criterion for failure under repeated loading, with a range of different amplitudes is Miner's Rule

$$\sum\left(\frac{n_i}{N_i}\right) = 1 \tag{2}$$

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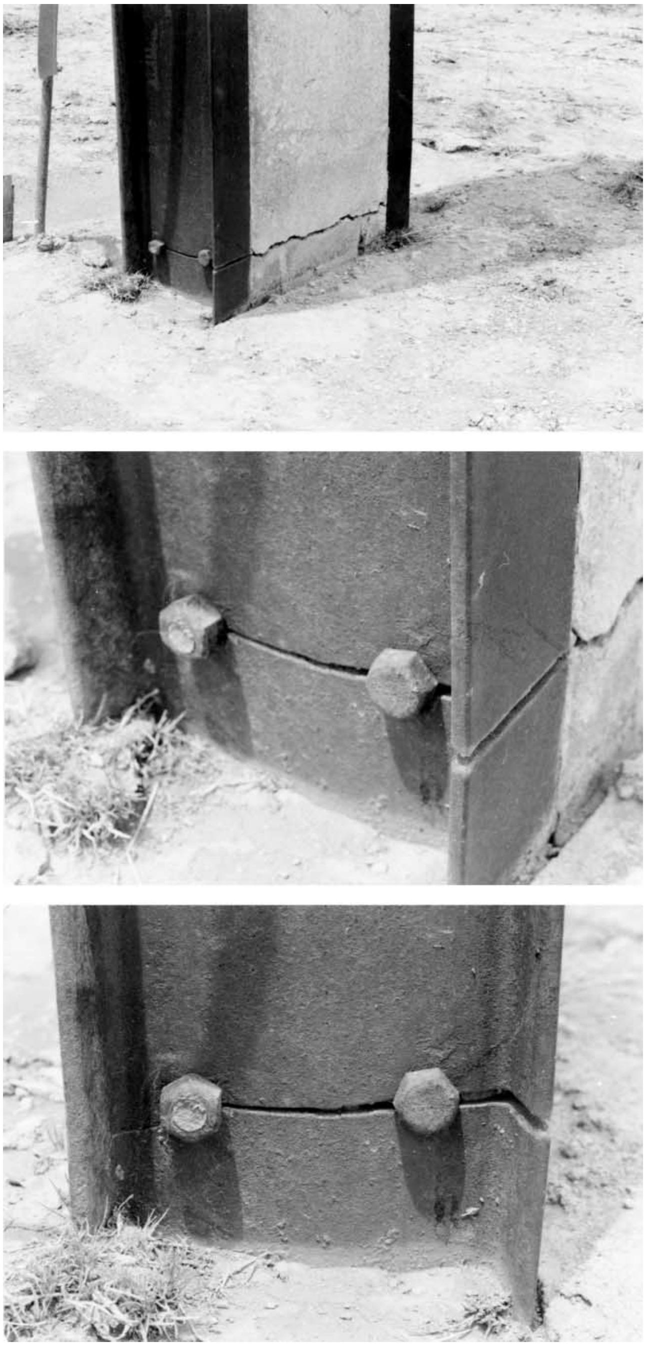

Fig. 1. An example of wind-induced fatigue failure of a steel utility pole.

where n_i is the number of stress cycles at an amplitude for which N_i cycles are required to cause failure. Thus failure is expected when the sum of the fractional damage for all stress levels is unity.

Note that there is no restriction on the *order* in which the various stress amplitudes are applied in Miner's Rule. Thus we may apply it to a random loading process which can be considered as a series of cycles with randomly varying amplitudes.

3. Narrow band fatigue loading

Some wind loading situations produce resonant 'narrow-band' vibrations (Fig. 2). For example, the along-wind response of structures with low natural frequencies, and cross-wind vortex induced response of circular cylindrical structures with low damping. In these cases, the resulting stress variations can be regarded as quasi-sinusoidal with randomly varying amplitudes.

For a narrow-band random stress $s(t)$, the proportion of cycles with amplitudes in the range from s to $s+\delta s$, is $f_p(s)$. δs, where $f_p(s)$ is the probability density of the peaks. The total number of cycles in a time period, T, is ν_o^+T, where ν_o^+ is the rate of crossing of the mean stress. For narrow band resonant vibration, ν_o^+ may be taken to be equal to the natural frequency of vibration.

Then the total number of cycles with amplitudes in the range s to δs

$$n(s)=\nu_o^+Tf_p(s)\,\delta s. \tag{3}$$

If $N(s)$ is the number of cycles at amplitude s to cause failure, then the fractional damage at this stress level

$$=\frac{n(s)}{N(s)}=\frac{\nu_o^+Tf_p(s)s^m\,\delta s}{K}$$

where Eq. (3) has been used for $n(s)$, and Eq. (1) for $N(s)$.

The total expected fractional damage over all stress amplitudes is then, by Miner's Rule:

$$D=\sum_0^\infty\frac{n(s)}{N(s)}=\frac{\nu_o^+T\int_0^\infty f_p(s)s^m\,\mathrm{d}s}{K} \tag{4}$$

Wind-induced narrow-band vibrations can be taken to have a normal or Gaussian probability distribution. If this is the case, then the peaks or amplitudes, s, have a Rayleigh distribution (e.g. Crandall and Mark [2]):

$$f_p(s)=\frac{s}{\sigma^2}\exp\left(-\frac{s^2}{2\sigma^2}\right) \tag{5}$$

where σ is the standard deviation of the entire stress history. Derivation of Eq. (5) is based on the well-known level crossing formula of Rice [3].

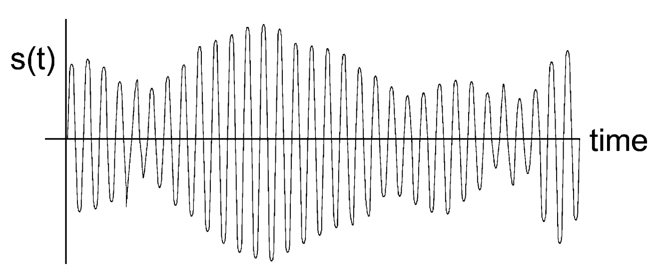

Fig. 2. Narrow-band random vibration.

Substituting into Eq. (4)

$$D=\frac{\nu_o^+T}{K\sigma^2}\int_0^\infty s^{m+1}\exp\left(-\frac{s^2}{2\sigma^2}\right)ds=\frac{\nu_o^+T}{K}(\sqrt{2}\sigma)^m\Gamma\left(\frac{m}{2}+1\right) \quad (6)$$

Here the following mathematical result has been used (Dwight [4]):

$$\int_0^\infty x^q\exp[-(rx)^p]dx=\frac{1}{pr^{q+1}}\Gamma\left(\frac{q+1}{p}\right) \quad (7)$$

where $\Gamma(x)$ is the Gamma function.

Eq. (6) is a very useful 'closed-form' result, but it is restricted by two important assumptions:

- 'high-cycle' fatigue behaviour in which steel is in the elastic range, and for which an *s–N* curve of the form of Eq. (1) is valid, has been assumed;
- narrow band vibration in a single resonant mode of the form shown in Fig. 2 has been assumed. In wind loading this is a good model of the behaviour for vortex-shedding induced vibrations in low turbulence conditions. For along-wind loading, the background (sub-resonant) components are almost always important, and result in a random wide-band response of the structure.

4. Wide-band fatigue loading

Wide-band random vibration consists of contributions over a broad range of frequencies, with a large background response peak — this type of response is typical for wind loading (Fig. 3). A number of cycle counting counting methods for wide-band stress variations have been proposed. One of the most realistic of these is the 'rainflow' method proposed by Matsuishi and Endo [5]. In this method, which uses the analogy of rain flowing over the undulations of a roof, cycles associated with complete hysteresis cycles of the metal, are identified. Use of this method rather than a simple level-crossing approach, which is the basis of the narrow-band approach described in the previous section, invariably results in fewer cycle counts.

A useful empirical approach has been proposed by Wirsching and Light [6]. They proposed that the fractional fatigue damage under a wide-band random stress variation can be written as:

$$D=\lambda D_{nb} \quad (8)$$

where, D_{nb} is the damage calculated for narrow-band vibration with the same standard deviation, σ [Eq. (6)]. λ is a parameter determined empirically. The approach used to determine λ was to use simulations of wide-band processes with spectral densities of various shapes and bandwidths, and rainflow counting for fatigue cycles.

The formula proposed by Wirsching and Light to estimate λ was:

$$\lambda=a+(1-a)(1-\varepsilon)^b \quad (9)$$

where a and b are functions of the exponent m [Eq. (1)], obtained by least-squares fitting, as follows:

$$a\cong 0.926-0.033m \quad (10)$$

$$b\cong 1.587m-2.323 \quad (11)$$

ε is a spectral bandwidth parameter equal to:

$$\varepsilon=1-\frac{m_2^2}{m_0m_4} \quad (12)$$

where m_k is the kth moment of the spectral density defined by:

$$m_k=\int_0^\infty f^kS(f)df \quad (13)$$

For narrow band vibration ϵ tends to zero, and, from Eq. (9), λ approaches 1. As ϵ tends to its maximum possible value of 1, λ approaches a, given by Eq. (10). These values enable upper and lower limits on the damage to be determined.

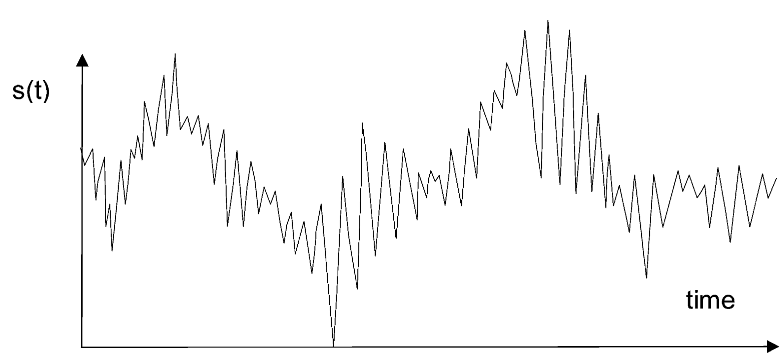

Fig. 3. Wide-band random vibration.

5. Application to wind loading

5.1. Effect of varying wind speed

Eq. (6) applies to a particular standard deviation of stress, σ, which in turn is a function of mean wind speed, $\bar{U}$. This relationship can be written in the form:

$$\sigma = A\bar{U}^n \tag{14}$$

where n is a power between 2 and about 2.5, with the higher values applying if resonant response is significant.

The constant A can be estimated from a static structural analysis and from the longitudinal turbulence intensity, if a quasi-steady model of wind loading is assumed, and the background wind loading is dominant. Then, A is approximately equal to $2CI_u$, where the mean stress is equal to $C\bar{U}^2$, and I_u is the longitudinal turbulence intensity. The background response will dominate at lower values of the mean wind speed, $\bar{U}$. As the mean wind speed increases the resonant component will become more significant. This will be reflected mainly in the value of the exponent, n, but will also influence the value for A.

The mean wind speed, $\bar{U}$, itself, is a random variable. Its probability distribution can be represented by a Weibull distribution:

$$f_U(\bar{U}) = \frac{k\bar{U}^{k-1}}{c^k}\exp\left[-\left(\frac{\bar{U}}{c}\right)^k\right] \tag{15}$$

where k is a shape factor, and c is a scale factor.

When k is equal to 2, the Weibull distribution is known as the Rayleigh distribution. In Fig. 4, some recorded wind data has been fitted in two ways:

1. allowing the shape factor to vary;
2. fixing the shape factor to be equal to 2.

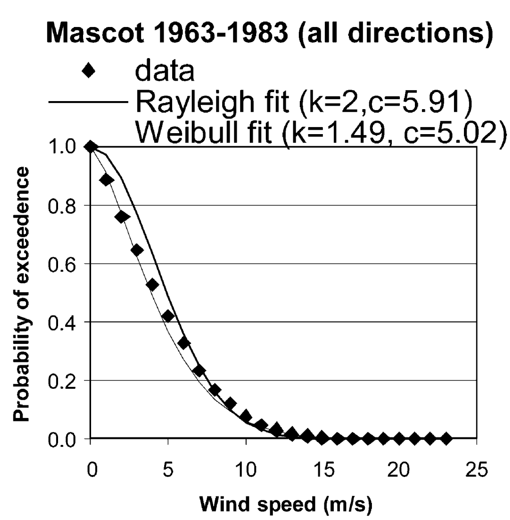

Fig. 4. Probability distributions of recorded mean wind speed at Sydney, Australia.

The total damage from narrow-band vibration for all possible mean wind speeds is obtained from Eqs. (6), (14) and (15) and integrating.

The fraction of the time T during which the mean wind speed falls between U and $U+\delta U$ is $f_U(U)\delta U$.

Hence the amount of damage generated while this range of wind speed occurs is from Eqs. (6) and (14):

$$D_U = \frac{\nu_o^+ T f_U(U)\delta U}{K}\left(\sqrt{2}AU^n\right)^m \Gamma\left(\frac{m}{2}+1\right)$$

The total damage in time T during all mean wind speeds between 0 and ∞

$$D = \frac{\nu_o^+ T\left(\sqrt{2}A\right)^m}{K}\Gamma\left(\frac{m}{2}+1\right)\int_0^\infty U^{mn} f_U(U)\,\mathrm{d}U =$$

$$\frac{\nu_o^+ T\left(\sqrt{2}A\right)^m}{K}\Gamma\left(\frac{m}{2}+1\right)\int_0^\infty U^{mn+k-1}\frac{k}{c^k}\exp\left[-\left(\frac{U}{c}\right)^k\right]\mathrm{d}U.$$

This is now of the form of Eq. (7), so that:

$$D = \frac{k\nu_o^+ T\left(\sqrt{2}A\right)^m}{Kc^k}\Gamma\left(\frac{m}{2}+1\right)\frac{c^{mn+k}}{k}\Gamma\left(\frac{mn+k}{k}\right) \tag{16}$$

$$= \frac{\nu_o^+ T\left(\sqrt{2}A\right)^m c^{mn}}{K}\Gamma\left(\frac{m}{2}+1\right)\Gamma\left(\frac{mn+k}{k}\right)$$

This is a useful closed-form expression for the fatigue damage over a lifetime of wind speeds, assuming narrow band vibration.

For wide-band vibration, the same approach can be adopted, making use of the results of Section 2 to give upper and lower limits on the damage.

For wide-band vibration, Eq. (16) can be modified, following Eq. (8), to:

$$D = \frac{\lambda\nu_o^+ T\left(\sqrt{2}A\right)^m c^{mn}}{K}\Gamma\left(\frac{m}{2}+1\right)\Gamma\left(\frac{mn+k}{k}\right) \tag{17}$$

By setting D equal to 1 in Eqs. (16) and (17), we can obtain lower and upper limits to the fatigue life as follows:

$$T_{\text{lower}} = \frac{K}{\nu_o^+\left(\sqrt{2}A\right)^m c^{mn}\Gamma\left(\frac{m}{2}+1\right)\Gamma\left(\frac{mn+k}{k}\right)} \tag{18}$$

$$T_{\text{upper}} = \frac{K}{\lambda\nu_o^+\left(\sqrt{2}A\right)^m c^{mn}\Gamma\left(\frac{m}{2}+1\right)\Gamma\left(\frac{mn+k}{k}\right)} \tag{19}$$

For the background response, the crossing rate, ν_o^+, is itself a weak function of mean wind speed, and should not strictly be treated as a constant. However, this is a second-order effect, and for practical purposes, it should be adequate to take it as constant, with a conservative value, say one half the lowest structural natural frequency.

5.2. Effect of wind direction

The effect of wind direction can be incorporated by fitting a Weibull distribution to wind speeds from each direction sector, i:

$$f_U(\bar{U}_i)=\frac{k\bar{U}_i}{c_i^2}\exp\left[-\left(\frac{\bar{U}_i}{c_i}\right)^k\right]. \tag{20}$$

The structural response will also be a function of wind direction:

$$\sigma_i=A\bar{U}_i^n. \tag{21}$$

So that the damage generated from winds from direction, i, is given by:

$$D_i=\frac{\lambda\nu_o^+T\left(\sqrt{2}A_i\right)^m c_i^{mn}}{K}\Gamma\left(\frac{m}{2}+1\right)\Gamma\left(\frac{mn+k}{k}\right)$$

so that the total damage for winds from N direction sectors is:

$$D=\sum_{i=1}^{N}p(\theta_i)D_i= \tag{22}$$

$$\frac{\lambda\nu_o^+T}{K}\Gamma\left(\frac{m}{2}+1\right)\Gamma\left(\frac{mn+k}{k}\right)\sum_{i=1}^{N}p(\theta_i)\left(\sqrt{2}A_i\right)^m c_i^{mn}$$

where $p(\theta_i)$ is the probability of the mean wind occurring within the direction sector i.

In practice, it is likely that only one or two wind directions will be important in the above summation.

6. Cycle count for narrow-band loading

The number of expected cycles exceeding various stress levels over a lifetime can be derived for narrow-band vibration in a similar manner to the derivation of fatigue life in the previous sections.

Thus from Eqs. (4) and (5), the number of cycles exceeding the stress level, s, per unit time is at a mean wind speed $\bar{U}$:

$$=\nu_o^+\exp\left(-\frac{s^2}{2\sigma^2}\right)=\nu_o^+\exp\left(-\frac{s^2}{2A^2\bar{U}^{2n}}\right) \tag{23}$$

from Eq. (14).

As before, the fraction of the time T during which the mean wind speed falls between U and $U+\delta U$ is $f_U(U)\delta U$, where $f_U(U)$ is given by the Weibull Distribution [Eq. (15)].

Then the number of cycles during the time T, exceeding the stress level, s, when the wind speed falls between U and $U+\delta U$, is:

$$=\nu_o^+T\frac{k\bar{U}^{k-1}}{c^k}\exp\left[-\left(\frac{\bar{U}}{c}\right)^k\right]\exp\left(-\frac{s^2}{2A^2\bar{U}^{2n}}\right)\delta\bar{U}$$

$$=\nu_o^+T\frac{k\bar{U}^{k-1}}{c^k}\exp\left[-\left(\frac{\bar{U}}{c}\right)^k-\left(\frac{s^2}{2A^2\bar{U}^{2n}}\right)\right]\delta\bar{U}$$

The total number of cycles exceeding s, in time T, for all mean wind speeds is then:

$$n(s)=\frac{k\nu_o^+T}{c^k}\int_0^\infty\bar{U}^{k-1}\exp\left[-\left(\frac{\bar{U}}{c}\right)^k-\left(\frac{s^2}{2A^2\bar{U}^{2n}}\right)\right]d\bar{U} \tag{24}$$

Eq. (24) apparently must be integrated numerically. It will overestimate the number of fatigue cycles for wide-band vibration, and is therefore conservative for wind loading.

7. Example

As an example of calculation of fatigue life using Eqs. (18) and (19), assume the following values:

$m=5$; $n=2$; $k=1.5$; $\nu_0^+=0.5$ Hertz;

$K=2\times10^{15}[\text{MPa}]^{1/5}$; $c=6$ m/s; $A=0.1\frac{\text{MPa}}{(\text{m/s})^2}$

$$\Gamma\left(\frac{m}{2}+1\right)=\Gamma(3.5)=e^{1.201}=3.32\ \Gamma\left(\frac{mn+k}{k}\right)=\Gamma(7.667)$$

$$=e^{7.8608}=2594$$

Then from Eq. (18)

$$T_{\text{lower}}=\frac{2\times10^{15}}{0.5\times\left(\sqrt{2}\times0.1\right)^5\times6^{10}\times3.32\times2594}=1.357\times10^8\text{ s}$$

$$=\frac{1.357\times10^8}{365\times24\times3600}\text{ years}=4.3\text{ years}$$

From Eq. (10), $a=0.926-0.033\,m=0.761$.

From Eq. (9), this is a lower limit for λ

$$T_{\text{upper}}=\frac{T_{\text{lower}}}{\lambda}=\frac{4.3}{0.761}\text{ years}=5.7\text{ years}$$

This example illustrates the sensitivity of the estimates of fatigue life to the values of both A and c. For example, increasing A to 0.15 MPa/[(m/s)^2] would decrease the fatigue life by 7.6 times (1.5^5). Decreasing c from 6 to 5 m/s will increase the fatigue life by 6.2 times $(6/5)^{10}$.

8. Conclusions

Useful closed-form expressions for estimating the upper and lower limits of fatigue life of structures, or elements of structures, under along-wind loading, have been derived. Some assumptions have been made to derive these expressions — the principal one being that the mean wind speed follows a Weibull probability distribution. Results by Wirsching and Light [6] based on numerical simulations have been used for wide-band vibration.

The use of closed-form expressions avoids the extensive numerical calculations that are usually required for this type of calculation, and is useful for approximate calculations to determine whether, in fact, fatigue under wind loading is a problem that needs to be considered for a particular structure. The theory developed in this paper is applicable to the accumulated fatigue damage from many storms of the temperate synoptic type. It may also be applicable to fatigue damage generated in a tropical cyclone event, provided the slowly varying 'mean' component can be fitted with a Weibull Distribution, and if the wind direction changes occurring during the passage of this type of storm, can be accommodated using the approach described in Section 5.2.

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